Physics 216/116

Lecture VI - Instabilities II

So far
- basic ideas of instability and relaxation
- R-T } interfacial instability → in depth.
  K-H }
- includes discussion of waves, surface tension, etc.

Now
- More on K-H

then
- Convection (R-B) - distributed vorticity
  - ideal physics
    ~ Schwarzschild criterion
  - R-B physical picture
    ~ Rayleigh number Ra.
  - R-B eqns
  - R-B threshold ⇒ Ra crit
  - discussion
- Rotating Convection ⇒ { discussion to Lecture VII with rotation
  - freezing-in law,
  - Taylor-Proudman Theorem, implications

- rotating convection (relate HW)

$$\omega^2 = \frac{k_H^2}{k^2} g \frac{\partial S}{\partial z} + \frac{k_V^2 4\Omega^2}{k^2}$$

→ Taylor columns, Proudman pillars.

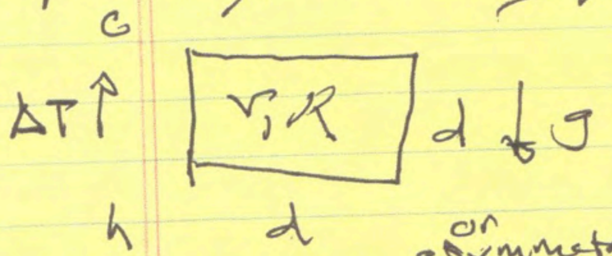
- physics of inertial waves *
- relation to magnetic field, magneto-convection

→ Convection (Rayleigh-Benard)

~ intensively studied → thousands of papers

~ transport of heat in astrophysical bodies, atmosphere, plasma confinement, etc.

~ prototype: Rayleigh-Benard problem



~ no slip b.c.

~ variants on B.C.

or asymmetric

- critical ΔT, or Ra $(\sim \Delta T)$, for instability?
- pattern structure?

variant: box rotates, often fast.

What is effect of rotation on convection?

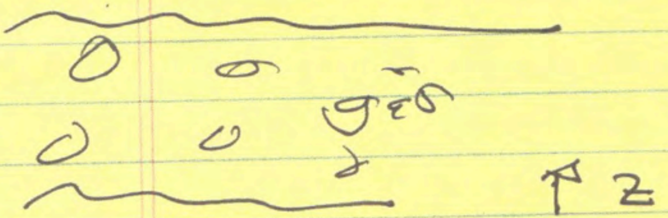
(connects to aim of Module II).

→ Ideal Fluid, ∞ Medium → $\left\{ \text{Schwarzschild Criterion} \right.$

i.e. stellar atmosphere

$\Downarrow$

$\left\{ \text{onset convection in astrophysical systems} \right.$



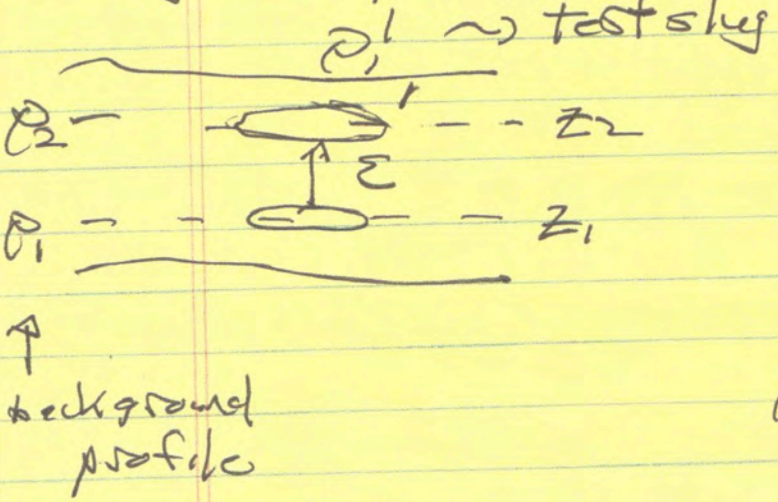
$\frac{d\rho}{dz} < 0$

$\frac{dP}{dz} < 0$ R-T stable

$$\frac{dP}{dz} = -\rho g \quad (g>0)$$

$P\rho^{-\gamma} \simeq \text{const}$ - eqn. state

As basic idea of convection, consider the virtual displacement of a slug/blob of gas upward.



$\rho_1' > \rho_2$ →

blob sinks, system stable

$\rho_2' < \rho_2$ → blob buoyant, rises.

For infinitesimal displacement,

$\Sigma \sim \Delta z$

$$\rho_2 = \rho_1 + \frac{\partial \rho}{\partial z} \Delta z$$

For ρ_1' → system obeys equation of state

$$P \rho^{-\gamma} = \text{const.}$$

→ displaced blob (i.e. ①')
$\delta p \sim 0$ → rapidly comes to pressure equilibration with s

blob

$$\boxed{\frac{\Delta z}{c_s} << \tau_{Rise}} \Leftrightarrow v << c_s$$

↳ nearly incompressible

∴

$$p_1' = p_2 = p_1 + \Delta z \frac{dp_1}{dz}$$

So

$$p_1 \rho_1^{-\gamma} = p_1' \rho_1'^{-\gamma}$$

$$P_1 \rho_1^{-\gamma} = \overset{P_2}{\left(P_1 + \Delta z \frac{dP_1}{dz}\right)} \rho_1'^{-\gamma}$$

so

$$\left(\frac{\rho_1'}{\rho_1}\right)^{\gamma} = 1 + \frac{\Delta z}{P_1} \frac{dP_1}{dz}$$

$$\left(\rho_1'/\rho_1\right) = \left(1 + \frac{\Delta z}{P_1} \frac{dP_1}{dz}\right)^{1/\gamma}$$

$$\simeq 1 + \frac{\Delta z}{\gamma} \frac{1}{P_1} \frac{dP_1}{dz}$$

$$\left(\rho_2/\rho_1\right) = 1 + \frac{1}{\rho_1} \Delta z \frac{d\rho_1}{dz}$$

so blob buoyant if — (unstable)

$$\frac{\rho_1'}{\rho_1} < \frac{\rho_2}{\rho_1} \Rightarrow \frac{\Delta z}{\gamma} \frac{1}{P_1} \frac{dP_1}{dz} < \frac{\Delta z}{\rho_1} \frac{d\rho_1}{dz}$$

$$\Rightarrow \boxed{\frac{1}{\gamma} \frac{1}{P_1} \frac{dP_1}{dz} < \frac{1}{\rho_1} \frac{d\rho_1}{dz}}$$

or, as both gradients negative:

$$\frac{1}{\gamma}\left|\frac{1}{P_1}\frac{dP_1}{dz}\right| > \frac{1}{\rho_1}\left|\frac{d\rho}{dz}\right|$$

and as $S' = c\ln(P\rho^{-\gamma})$

$$\frac{dS}{dz} = \frac{1}{P}\frac{dP}{dz} - \frac{\gamma}{\rho}\frac{d\rho}{dz}$$

$\Rightarrow$ buoyant blob - instability - if:

$\frac{dS}{dz} < 0$ → "superadiabatically stratified"

stable - if: (blob sinks)

$\frac{dS}{dz} > 0$ → "subadiabatically stratified"

$\frac{dS}{dz} = 0$ → adiabatically stratified

Schwarzschild criterion for convection instability:

$$\boxed{\frac{dS}{dz} < 0}$$

or

$$\boxed{\frac{1}{P}\frac{dP}{dz} < \frac{\gamma}{\rho}\frac{d\rho}{dz}}$$

free energy criterion

or $P = k_b \rho T$

$$\frac{1}{\rho}\frac{d\rho}{dz} + \frac{1}{T}\frac{dT}{dz} < \frac{\gamma}{\rho}\frac{d\rho}{dz}$$

$$\Rightarrow \boxed{\frac{1}{T}\frac{dT}{dz} < \frac{(\gamma-1)}{\rho}\frac{d\rho}{dz}}$$

i.e. 'sufficiently steep' temperature gradient, rel. to density

'sufficient' → $\gamma - 1$

$\gamma \equiv$ captures essential thermal properties

↓

E.O.S.

Note: Convenient to work with S in full analysis, as (in ideal fluid)

$$\frac{\partial S}{\partial t} + \mathbf{v} \cdot \nabla S = 0$$

(isentropic dynamics)

↳ fluctuation

$$S = \langle S \rangle + \tilde{S}$$

↳ mean profile

and,

$$\frac{\partial \tilde{S}}{\partial t} + \tilde{v} \frac{\partial \langle S \rangle}{\partial z} = 0$$

Now

$$S \sim \ln\left(P \rho^{-\gamma}\right)$$

$$\sim \ln\left(T \rho^{-(\gamma-1)}\right)$$

$$\tilde{S} = \frac{1}{T \rho^{-(\gamma-1)}} \left[\hat{T} \rho^{-(\gamma-1)} + T\left(-(\gamma-1)\right) \rho^{-\gamma} \tilde{\rho} \right]$$

$$\tilde{S} = \left[\frac{\hat{T}}{T} - (\gamma-1) \frac{\tilde{\rho}}{\rho} \right]$$

Anticipate: $\nabla \cdot \underline{v} = 0$ so no sound wave in convection dynamics

slow rise

$$\delta P = 0 \Rightarrow \delta\rho \, T = -\delta T \, \rho$$

$$\boxed{\tilde{\rho}/\rho = -\frac{\tilde{T}}{T}}$$

→ relates buoyancy to temperature.

Check: $\frac{\partial \underline{v}}{\partial t} = -\frac{\underline{\nabla} p}{\rho} + \underline{g}$

$\underline{\nabla} \cdot \tilde{\underline{v}} \Rightarrow$

$\partial_t \underline{\nabla} \cdot \tilde{\underline{v}} = -\frac{\nabla^2 \tilde{p}}{\rho} + \cancel{\nabla g}$

$\underline{\nabla} \cdot \tilde{\underline{v}} = 0 \quad \Rightarrow \quad \nabla^2 \tilde{p} = 0$

$$k^2 \tilde{p}_{\underline{k}} = 0$$

so

$$\frac{\tilde{\rho}_k}{\rho} = -\frac{\tilde{T}_k}{T}$$

N.B.: Essence of "Boussinesq, incompressible" convection is:

- dynamics slow, relative to sound wave
- vertical wave vector k s/t $k \underset{\text{scale height}}{L_{\rho,z}} \gg 1$.

so,

$$\overset{\textcircled{1}}{\frac{\partial\tilde{\rho}}{\partial t}} + \overset{\textcircled{2}}{\tilde{v}_z\frac{d\rho_0}{dz}} + \overset{\textcircled{3}}{\rho_0\,\underline{\nabla}\cdot\tilde{\underline{V}}} = 0$$

$$\begin{array}{ccc} \wr & \wr & \wr \\ \frac{1}{\tau}\frac{\tilde{\rho}}{\rho_0} & \frac{\tilde{v}_z}{L_\rho} & \rho_0 k_z \tilde{V} \end{array}$$

Take $L_\rho \sim L_s$

→ $\tau \gg (kc_s)^{-1}$ ⇒ drop ①

→ $k_z L_s \gg 1$ ⇒ drop ②

$\underline{\nabla}\cdot\tilde{\underline{V}} = 0$ emerges as effective condition.

→ simplest subsonic extension is:

$\nabla\cdot(\rho\underline{V}) = 0$ → incompressible mass flow.

(so called "anelastic eqn.")

$$\underline{\nabla}\cdot\tilde{\underline{V}} + \frac{\tilde{v}_z}{\rho}\frac{d\rho}{dz} = 0.$$

→ decouples sound wave

and modify freezing-in law. → retains finite scale height.

⊗

$\tilde{S} = \gamma \frac{\tilde{T}}{T}$ ⟺ entropy fluctuation tied directly to temperature fluctuation.

Now, at level of estimation: For buoyancy time scale.

$$\frac{\partial \tilde{v}_z}{\partial t} = -\frac{\partial_z \tilde{P}}{\rho_0} - g\frac{\tilde{\rho}}{\rho_0}\hat{z}$$

$$= g\frac{\tilde{T}}{T_0}\hat{z}$$

$$\frac{\partial \tilde{T}/T_0}{\partial t} = -\tilde{v}_z \frac{dS_0}{\partial z}$$

$$= \frac{1}{\gamma} g \frac{\partial S_0}{\partial z}\tilde{v}_z \hat{z}$$

τ_b → buoyancy time scale

$$\frac{\tilde{v}_z}{\tau_b^2} \sim \frac{1}{\gamma} g \frac{\partial S_0}{\partial z}\tilde{v}_z$$

gives:

$$\boxed{1/\tau_b^2 \sim \frac{g}{\gamma}\frac{\partial S_0}{\partial z}}$$

If entertain <u>now</u> <u>dissipation</u>:

→ viscosity, i.e. momentum diffusion:

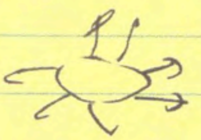
$$\partial_t \underline{\tilde{v}} \to \partial_t \underline{\tilde{v}} - \nu \nabla^2 \underline{\tilde{v}}$$

smears out rise motion.

and $1/\tau_\nu \sim \frac{\nu}{\ell^2}$ → specifies viscous time on scale ℓ

→ thermal diffusivity

heat lost from blob



$$\partial_t \tilde{T} \to \partial_t \tilde{T} - \underset{\substack{\downarrow \\ \text{thermal conduction}}}{\kappa} \nabla^2 \tilde{T}$$

and

$1/\tau_\kappa \sim \kappa/\ell^2$ → specifies thermal diffusion time on scale ℓ

And can then note:

Diffusion effects will limit buoyancy → i.e. smear out heat parcel (moving) if

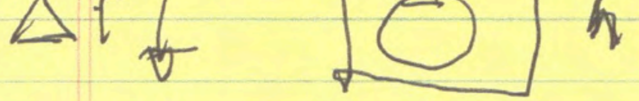
$$1/\tau_b^2 \sim 1/\tau_\nu \tau_\kappa$$

point → instability needs free energy sufficient to overcome dissipation so, taking $\partial S/\partial z$

$$\boxed{\frac{T_\nu T_K}{T_b^2} \sim g \frac{\partial\langle S\rangle}{\partial z} l^4/\nu K \equiv \underset{\text{Rayleigh number.}}{\underset{\downarrow}{Ra}}}$$

↳ 2nd most popular dimensionless # in fluid mechanics.

For convective fluid in a box:



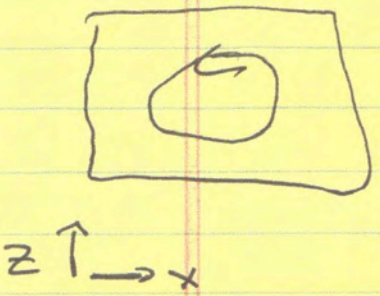
$$\underset{\text{coefficient of thermal expansion}}{\delta\rho = -\underset{\downarrow}{\alpha}\,\delta T}$$

$$\boxed{Ra \equiv g\,\Delta T\alpha h^3/\nu K}$$

Clearly need $Ra > \# \sim 1$ for convective instability to occur.

Point of analysis is to determine $(Ra)_{crit}$.

Some calculation



- Consider vorticity $\perp$ to x, z
  $\Rightarrow \omega_y$.

- $\underline{v} = \underline{\nabla}\phi \times \hat{y}$

then $\underline{\omega}_y = -\partial_x^2\phi - \partial_z^2\phi$

Linearized ideal eqns:

$$\frac{\partial \underline{\tilde{v}}}{\partial t} = -\frac{\underline{\nabla}\tilde{p}}{\rho} - g\tilde{\rho}\hat{z}$$

$\tilde{\rho}$ fluctuation drives convection

eqbrm.

i.e.

$$\cancelto{0}{\frac{\partial \underline{v}_0}{\partial t}} + \frac{\partial \underline{\tilde{v}}}{\partial t} = -\cancel{\frac{\underline{\nabla}p_0}{\rho_0}} - \frac{\underline{\nabla}\tilde{p}}{\rho_0} - \cancel{g\rho_0\hat{z}} - g\frac{\tilde{\rho}}{\rho_0}\hat{z}$$

hydrostatic eqbrm.

Also, "Boussinesq Approximation"
(consistent with $\underline{\nabla}\cdot\underline{V} = 0$)

$\Rightarrow$

- treat ρ as constant, except where deviation $\tilde{\rho}$ is compared to zero.

- $\tilde{\rho}$ only in buoyancy force

so

$$\frac{\partial \underline{v}}{\partial t} = -\frac{\nabla \tilde{P}}{\rho} - g\frac{\tilde{\rho}}{\rho_0}\hat{z}$$

$$= -\frac{\nabla \tilde{P}}{\rho_0} + g\frac{\tilde{T}}{T_0}\hat{z}$$

$\hat{y} \cdot \nabla \times \Rightarrow$

$$\frac{\partial}{\partial t}\left(-\nabla^2 \phi\right) = g\frac{\partial}{\partial x}\left(\frac{\tilde{T}}{T_0}\right) - \nu\nabla^2\left(-\nabla^2\phi\right)$$

$\nabla^2 = \partial_x^2 + \partial_z^2$

$$\frac{\partial}{\partial t}\left(\frac{\delta \tilde{T}}{T_0}\right) = -v_z \frac{d\langle s\rangle}{dz} + \chi \nabla^2 T$$

→ convection roll equations, with B.C. and dissipation effects

→ ideal: $\omega^2 = g(\partial s/\partial z)\frac{k_x^2}{k^2}$

c.f. → Chandra. → Chapt. 2

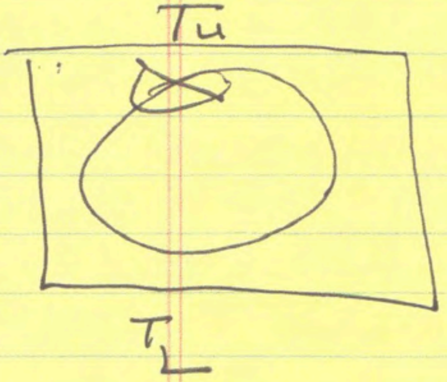
Manneville → chapt. 3,4
[Dissipative Structures and Weak Turbulence]

Now, universal notation for problem given by Chandrasekhar so switch:

$$W \rightarrow \hat{V}_z$$

$$\Theta \rightarrow \hat{T}/T \qquad \text{and} \qquad \left\{ \begin{array}{l} \delta\rho = -\alpha\, \delta T \\ \beta \equiv -\dfrac{dT}{dz} = \dfrac{\Delta T}{h} \end{array} \right.$$

$$\omega \rightarrow \omega_z$$



i.e. $\left\{ \nabla^2 T = 0 \right.$ for eqbrm. → linear vertically with b.c. fixed

$\hat{z}\cdot(\nabla\times\nabla\times \text{NSE})$

$$\boxed{\begin{array}{l} \dfrac{\partial}{\partial t} \nabla^2 W = g\alpha \overset{\nabla_h^2 \text{ before.}}{\left(\dfrac{\partial^2\Theta}{\partial x^2} + \dfrac{\partial^2\Theta}{\partial y^2} \right)} + \nu \nabla^4 W \\ \\ \dfrac{\partial \Theta}{\partial t} = \beta W + \kappa \nabla^2 \Theta \end{array}}$$

same, in slightly different notation.

n.b. $V_z = \partial_x \phi \rightarrow i k_x \phi$

Now, $T_0 = T_b - \beta z$

$\beta = \Delta T / h$.

Can de-dimensionalize:

length → h

time → h^2/K

$h/T_K \to \underline{V}$

$K\nu/\alpha g h^3 \to T$

$K^2/h^2 \to P/\rho_0$

so

$$\boxed{\begin{aligned} \partial_t \nabla^2 w &= P \nabla^4 w + \nabla_h^2 \theta \\ \partial_t \theta &= \nabla^2 \theta + R_a w \end{aligned}}$$

2 parameters specify system:

$R_a = g \alpha \beta h^3 / \nu K$

$P = \nu / K$ → Prandtl #

↳ relative strength of dissipations.

↳ another key dimless #.

Game Now becomes;

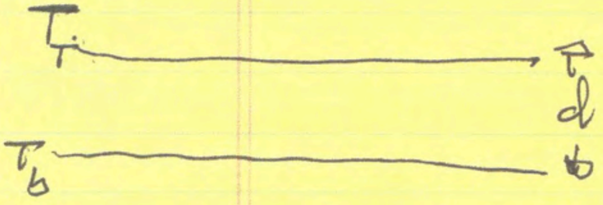
— compute Ra_{crit} for onset,

for

— P value given (~ 1 here)
— given $k_y \to k_x \Rightarrow$ scanned.
— boundary conditions. ⇒ the lesson.

⇒ Task is laborious.

Lesson: Boundary conditions set effect/behavior of dissipation. ✓



wide tank, don't worry about lateral B.C.

$\tilde{\Theta} = 0$ at $z = 0, h$.

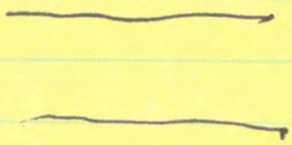
$w = v_z = 0$ at $z = 0, h$.

For other B.C., can envision two scenarios (of several).

① → no-slip

② → stress free (Rayleigh — 2 free boundaries)
(1916)

① For no slip: (rigid)



$$-\ \tilde{v}_z\Big|_{0,h} = \tilde{w}\Big|_{0,h} = 0$$

$$\tilde{v}_h\Big|_{0,h} = 0$$

but worked with w: $\underline{\underline{\infty}}$

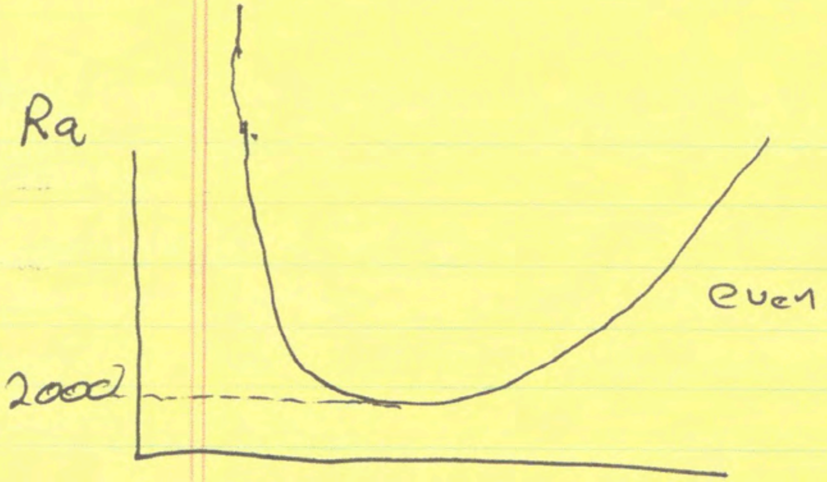
$$\nabla_h \tilde{v}_h + \partial_z \tilde{v}_z = 0$$

as all ∇_h of $\underline{\tilde{v}}$ vanish → i.e. all horizontal derivatives of $\underline{V}$ vanish as $\underline{V}$ vanishes.

$$\nabla_h \tilde{v}_h = 0 \quad \text{so} \quad \underline{\partial_z \tilde{v}_z = 0}$$

$$\boxed{\partial_z W = 0.}$$

⇒



$\alpha = k_h h$
(norm. k_h)

Chandra.
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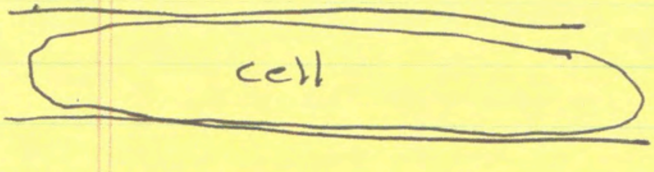
Ra crit ~ <u>2000</u>.

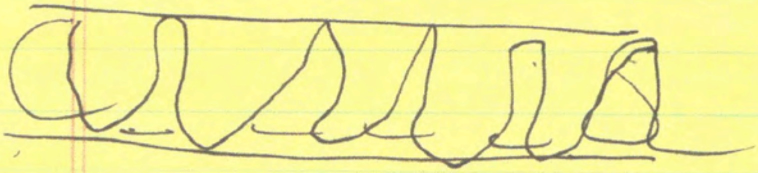
→ high k growth Ra_{crit} due rise
νk^2, etc → $\nu(k_v^2 + k_h^2)$

→ low k due



→ dissipn due
visc. effects
in upper, lower
layer

i.e. above, vs.



② For stress free:

open top (bottom):

$W\Big|_{0,h} = 0$, $\Theta\Big|_{0,h} = 0$

but free surface: no stress

$\underline{\underline{P}} = -\eta\, \partial_z V_h$ $\quad \underline{\underline{\quad}}$

shear stress delivered to surface → vanishes for free surface!

but have: $\partial_z V_h = 0$

and $\partial_h V_h = -\partial_z V_z$

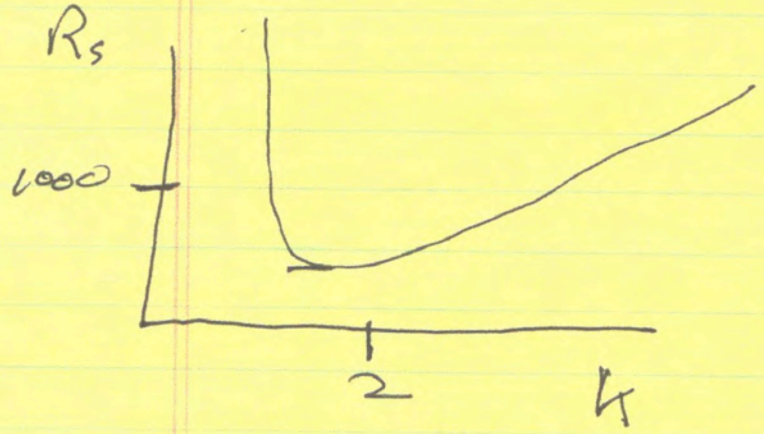
so $\underline{\partial_z \partial_h V_h = -\partial_z^2 V_z}$

$\partial_h \partial_z V_h = 0 = \underline{-\partial_z^2 V_z}$

So b.c. $\partial_z^2 W = 0 \Big|_{\text{top, bttm}}$

(replaces $\partial_z W = 0$ for no slip)

and have:



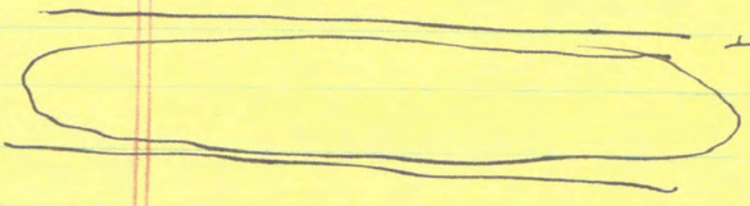
$Ra_{crit} \sim \frac{27\pi^4}{4}$ for $k_c = \pi/\sqrt{2}$

$\frac{27\pi^4}{4} \sim$

lower Ra_{crit}!

⇒ same material, but substantially low Ra_{crit} due stress free B.C in Ra_{crit}

again, low k region due:



→ layer dissipn.